Basics on commutative algebra and on category theory

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1 Categories

Definition 1.1. *A category $\mathcal{C}$ consists of:*

1. *A " collection" of objects: $ob(\mathcal{C})$;*
2. *$\forall X, Y \in Ob(\mathcal{C})$ a "collection" of morphisms : $Mor(X, Y)$ (morphism from X to Y), such that $\forall X, Y, Z \in ob(\mathcal{C})$*

$$\begin{array}{lcl} Mor(X,Y) \times Mor(Y,Z) & \to & Mor(X,Z) \\ (f,g) & \mapsto & g \circ f \end{array}$$

3. *associativity.*
4. *$\forall x \in ob(\phi)$, $\exists Id_X \in Mor(X, X)$ such that $\forall f : X \to Y$, $f \circ Id_X = f$, $Id_Y \circ f = f$.*

Example 1.2. 1. *<u>Sets</u>*
— *objects: sets*
— *$Mor(X, Y)$: maps from X to Y.*

2. *<u>Groups</u>*
— *objects: groups;*
— *$Mor(X, Y) = \{$ group homomorphism from X to $Y\}$*

3. *A ring, <u>A-Mod</u>*

Definition 1.3. 1. *A category $\mathcal{C}$ with $ob(\phi) =$ one element is called a monoid.*

2. *A category is locally small if $\forall X, Y \in ob(\mathcal{C})$, $Mor(Y, X)$ is a set. A category is small, if $ob(\mathcal{C})$, $Mor(X, Y)$ are sets.*
3. *$ob(\mathcal{C}) = \{X\}$, $Mor(X, X) = Id_X$ trivial monoid*
4. *Opposite category$=\mathcal{C}^{op}$ of a given category $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$ and $Mor_{\mathcal{C}^{op}}(X, Y) = Mor_{\mathcal{C}}(Y, X)$.*

Definition 1.4. 1. *$f : X \to Y$ is an isomorphism if $\exists g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$.*

Exercise 1.5. *Prove that such g is unique and that composition of isomorphisms is again an isomorphism.*

2. *$f : X \to Y$ is a monomorphism if $\forall Z$, $\forall\ Z \underset{h}{\overset{g}{\rightrightarrows}} x$ such that $f \circ g = f \circ h \Rightarrow g = h$.*

Exercise 1.6. *In sets, f is a monomorphism if and only if it is an injection.*

3. *$f : X \to Y$ is an epimorphism if $\forall Z \in ob(\mathcal{C})$, $\forall\ X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ such that $g \circ f = h \circ f \Rightarrow g = h$.*

Exercise 1.7.

1. *In sets: f is an isomorphism if and only if it is a bijection, f is a monomorphism if and only if it is an injection, and f is an epimorphism if and only if it is a surjection.*
2. *In any category, every isomorphism is both a monomorphism and an epimorphism. Is the converse true?*
3. *In groups and A-modules, prove that monomorphisms are precisely the injective homomorphisms and that isomorphisms are precisely the bijective homomorphisms.*
4. *Find an example of an epimorphism in rings that is not surjective.*
5. *Find an example of a morphism in topological spaces that is both a monomorphism and an epimorphism but not an isomorphism.*

1.1 Functors

Definition 1.8. *A functor $F : \mathcal{C} \to \mathcal{D}$ consists of*

1. *$F : ob(\mathcal{C}) \to ob(\mathcal{D})$,*
2. *$X, Y \in Ob(\mathcal{C})$, $F : Mor(X, Y) \to Mor(F(X), F(Y))$ such that $F(Id_X) = Id_{F(X)}$*
3. *$X \xrightarrow{f} Y \xrightarrow{g} Z$, $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ then $F(g \circ f) = F(g) \circ F(f)$*

Example 1.9.

1. *Identity functor $Id : \mathcal{C} \to \mathcal{C}$ and constant functors.*
2. *Forgetful functor*

$$\begin{array}{lcl} Groups & \to & Sets \\ X & \mapsto & X \\ Mor(X,Y) & \mapsto & Mor(X,Y) \end{array}$$

3. *$A \to B$ ring homomorphism*

$$\begin{array}{rcl} A - Mod & \to & B - Mod \\ M & \mapsto & M \otimes_A B \\ Mor(M,N) & \mapsto & Mor(M \otimes_A B, Y \otimes_A B) \\ u & \mapsto & u \otimes Id_B \end{array}$$

Definition 1.10. *A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C} \to \mathcal{D}^{op}$.*

Example 1.11. *$\mathcal{C}$ small category, $X \in ob(\mathcal{C})$,*

$$\begin{array}{lcl} h_X : \ \mathcal{C} & \to & Sets \\ Y & \mapsto & Mor(Y, X) \end{array}$$

—

$$\begin{array}{rcl} Mor(Y,Z) & \to & Mor(Mor(Z,X),Mor(Y,X)) \\ f: Y \to Z & \mapsto & g \mapsto g \circ f \end{array}$$

— $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{G}$, $G \circ F$ *composite associative*
— *Category of Categories:* $\underline{Cat}$ *object= categories and morphism= functor*

Exercise 1.12. *1. Define a covariant functor in sets that takes each set to its power set.*

2. Define a contravariant functor in sets that takes each set to its power set.

3. Prove that in the category of k-vector spaces, assigning dual spaces and transpose linear transformations is a contravariant functor.

Definition 1.13. $F : \mathcal{C} \to \mathcal{D}$ *functor*

1. F *is full if* $\forall X, Y \in ob(\mathcal{C})$, $Mor(X,Y) \twoheadrightarrow Mor(F(X),F(Y))$;
2. F *is faithful if* $Mor(X,Y) \hookrightarrow Mor(F(X),F(Y))$;
3. F *is fully faithful if it is full and faithful.*
4. *A subcategory* $\mathcal{C}$ *of* $\mathcal{D}$, $ob(\mathcal{C}) \subset ob(\mathcal{D})$,

$$X, Y \in Ob(\mathcal{C}), Mor_{\mathcal{C}}(X,Y) \subset Mor_{\mathcal{D}}(X,Y), Mor_{\mathcal{C}}(X,Y) \times Mor_{\mathcal{C}}(Y,Z) \to Mor_{\mathcal{C}}(X,Z)$$

5. *full subcategory* $\mathcal{C}$ *of* $\mathcal{D}$, $Mor_{\mathcal{C}}(X,Y) = Mor_{\mathcal{C}}(X,Y)$.

Example 1.14. — *group subcategory of Sets;*
— Mod_A *subcategory of Groups;*
— $Mod_{\mathbb{Q}}$ *full subcategory of Groups:* E, F $\mathbb{Q}$*-vector space,* $f : E \to F$ *homomorphism of groups*
$\mathbb{Z}$*-linear* $\Rightarrow f(x) = f(n/nx) = nf(1/nx)$, *so that* $1/nf(x) = f(1/nx)$.

1.2 Morphisms of functors

Definition 1.15. *1.*

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$$

functors, a morphism from F *to* G *(a natural transformation given by* $\forall X, Y \in Ob(\mathcal{C}), \alpha_X : F(X) \to G(X)$, $\alpha_X \in Mor(F(X),G(X))$ *such that* $\forall f : X \to Y$,

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

2. α *is an isomorphism if* α_X *is an isomorphism* $\forall X$.

3. *$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if $\exists G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F : \mathcal{C} \to \mathcal{C}$ is "naturally isomorphic" to Id_ϕ.*

Example 1.16. *1. V_K category of vector spaces over K, finite dimensional.*

$$V_K \to V_K$$

$ob(V_K) = \{(E, f) | E$ *finite dimensional vector space over* K, $Aut_K(E)\}$
$(E, f) \to (F, g)$

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "f"'] & F \arrow[d, "g"] \\ E \arrow[r, "h"] & F \end{tikzcd}$$

$$\begin{array}{rccc} F: & V_K & \to & V_K \\ & (E, f) & \mapsto & (E^*, (f^*)^{-1}) \end{array}$$

F is not isomorphic to Id_{V_K} (among not isomorphic to this)

2.

$$\begin{array}{rccc} G: & V_K & \to & V_K \\ & (E, f) & \mapsto & (E^{**}, (f^*)^*) \end{array}$$

G isomorphic to Id_{V_K}.

Definition 1.17. *$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ G is adjoint to F if $\forall X \in Ob(\mathcal{C}), \forall Y \in Ob(\mathcal{D})$,*

$$\alpha_{X,Y} : Mor_{\mathcal{D}}(F(X), Y) \simeq Mor_{\mathcal{D}}(X, G(Y))$$

natural bijection.

Example 1.18. *$A \to B$ ring hom,*

$$\underline{Mod}_A \xrightarrow{F} \underline{Mod}_B \xrightarrow{G} \underline{Mod}_A$$

$$M \longrightarrow M \otimes_A B$$

$$N \longrightarrow N$$

$$Hom_B(M \otimes_A B, N) \simeq Hom_A(M, N)$$

F is adjoint to G.

Exercise 1.19. *1. Prove the above adjoint situation with Hom and Id*

2. *Prove that the the free functor F that takes a set to the free group generated by its elements is adjoint to the forgetful functor G from groups to sets.*
3. *Let M be an A-module. Consider the functor F that tensors each A-module with M. Prove that this is an adjoint to the functor Hom that takes each A-module N to $Hom(M, N)$*

2 Presheaf

Definition 2.1. *X topological space. A pre sheaf $\mathcal{F}$ on X of groups is functor (contravariant) from $\underline{X}$ to Groups, where $\underline{X}$, $ob(\underline{X})$ = open subset of X.*
U, V open subsets of X

$$Mor(U,V) = \begin{cases} \text{canonical inclusion of } U \text{ in } V \text{ if } U \subset V \\ \emptyset \text{ otherwise} \end{cases}$$

$\forall U \subseteq X$ open subset

1. *$U \mapsto \mathcal{F}(U)$ group*
2. *if $U \subseteq V \Rightarrow \exists \rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ homomorphism of groups such that:*
   (a) *$f_{UU} = Id_{\mathcal{F}(U)}, \forall U$;*
   (b) *$U \subseteq V \subseteq W$ open subset*

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\rho_{WV}"] \arrow[d, "\rho_{WV}"'] & \mathcal{F}(U) \\ \mathcal{F}(V) \arrow[ur, "\rho_{VU}"'] & \end{tikzcd}$$

   (c) *$\mathcal{F}(\emptyset) = \{0\}$.*

Elements of $\mathcal{F}(V)$ are called a solution of $\mathcal{F}$ on V;
If $U \subseteq V$, $s \in \mathcal{F}(V)$, $s|_U := \rho_{VU}(s)$ is called the restriction of s to U.

Example 2.2. *X topological space $\forall U \subseteq X$, $\mathcal{F}(U) = C(U, \mathbb{R})$. If $U \subseteq V$, $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ restriction maps;*

Constant pre sheaf: we fixe a group G; $\mathcal{F} : U \to G$, $\emptyset \mapsto \{0\}$ (if $U \neq \emptyset$), $\rho_{UV} = Id_G$ if $V \neq \emptyset$ and $\rho_{U\emptyset} = 0$ maps is a presheaf.

2.1 Sheaves

Definition 2.3. *A sheaf on X is a pre sheaf $\mathcal{F}$ on X such that:*

1. *(Uniqueness condition) $\forall U \subseteq X$, $\forall \{U_i\}_i$ open covering of U,*
   $\begin{array}{ccc} \mathcal{F} & \to & \coprod_i \mathcal{F}(U_i) \\ s & \mapsto & (s|_{U_i})_i \end{array}$ *is injective.*
2. *(glueing condition): U, $\{U_i\}_i$ then $\forall (s_i)_i$, $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. There exists (unique) in $\mathcal{F}(U)$ such that $s|_{U_i} = s_i$, $\forall i$.*

Remarque 2.4. *1. Condition1. and 2.* $\Leftrightarrow$ $0 \longrightarrow \mathcal{F}(U) \xrightarrow{r} \coprod_i \mathcal{F}(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \coprod_{i,j} \mathcal{F}(U_i \cap U_j)$

$$s \longrightarrow (s|_{U_i})_i$$

$$\begin{cases} p((s_i)_i) & = & (s_{i(U_i \cap U_j)})_{(i,j)} \\ q((s_i)_i) & = & (s_j)_{U_i \cap U_j} \end{cases}$$

2. *exact* $\Leftrightarrow \begin{cases} r \text{ injective} \\ p(\alpha) = q(\alpha) \Rightarrow \alpha \in Im(r), \alpha \in \coprod_i \mathcal{F}(U_i) \end{cases}$.

$\mathcal{F}$ presheaf of group, therefore one can replace (p,q) by $(s_i)_i \mapsto (s_i)|_{U_i \cap U_j} \cdot (s_j)_{(U_i \cap U_j)^{-1}{}_{(i,j)}}$.

Example 2.5. *1. $U \mapsto C(U, \mathbb{R})$ is a sheaf;*

2. constant pre sheaf is not a sheaf because if $U \subseteq X$ not connected $U = U_1 \coprod U_2$, $G = G(U_1 \coprod U_2)$ so that $G = G(U_1)$ and $G = G(U_2)$. $\forall s \in G(U_1), t \in G(U_2)$, $s \neq t \Rightarrow s|_{U_1 \cap U_2} = t|_{U_1 \cap U_2}$ because $U_1 \cap U_2 = \emptyset$. But there is a section $h \in G(U)$ such that $h|_{U_1} = s$ and $h|_{U_2} = t$.
$X = \mathbb{C}$, $U \to \{f \in C^0(U, \mathbb{C}) | f(z) = \sum_{n \geqslant 0} a_n z^n$ absolutely convergent serie$\}$ presheaf not a sheaf.

2.2 Morphisms of pre sheaves

$\mathcal{F}$, $\mathcal{G}$ presheaves in X a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is given by: $\forall U set X$, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ homomorphism of groups such that $\forall V \subseteq U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

ϕ is natural transformation of functors. A morphism of sheaves is a morphisms of pre sheaves between sheaves.
We denote by $\underline{Psh}_X$ the category of pre sheaves on X and $\underline{Sh}_X$ the category of sheaves on X, it is a full subcategory $\underline{Psh}_X$.
$C^\infty(\mathbb{R})$ sheaf of C^∞ functions on X.

$$\begin{array}{ccc} C^\infty_X(\mathbb{R}) & \to & C^\infty_X(\mathbb{R}) \\ f & \mapsto & f' \end{array}$$

derivation, this is an homomorphism.

2.3 Subsection of a sheaf $\mathcal{F}$

A sub sheaf is a sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ subgroup $\forall \phi$.

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, phantom, "\subset"] \arrow[d, "\rho_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{G}(V) \arrow[r, phantom, "\subset"] & \mathcal{F}(V) \end{tikzcd}$$

2.4 Sheaf of ideals

— θ a sheaf of commutative unitary ring, sheaf of θ-modules

— $\forall U$
  — $\mathcal{F}(U)$ is a $\theta(U)$-module
  — $V \subseteq U$, $a \in \theta(V)$, $s \in \mathcal{F}(U) \Rightarrow (as)|_V = a|_V s|_V$
  — $\mathcal{F}$ is a sheaf.
— sheaf of ideals ρ of θ is sub sheaf of θ as sheaf of θ modules (i.e. $\rho(U)$ ideal of $\theta(U)$, $\forall U$).

Example 2.6. *1. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of groups then*

$$ker(\phi) : \begin{array}{lcl} U & \to & ker(\phi(U)) \\ \mathcal{F}(U) & \mapsto & \mathcal{G}(V) \end{array} \Rightarrow ker(\phi) \text{ is a sheaf}$$

What would be $Im(\phi)$, $\phi : \mathcal{F} \to \mathcal{G}$?
Natural way :

$$U \to Im(\phi(U)) : \mathcal{F}(U) \to \mathcal{G}(U)$$

This defines a sub pre sheaf. But in general it is not a sheaf. When $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

2. $\mathcal{H}$ = holomorphic function on $\mathbb{C}$. $U \subseteq \mathbb{C}$, $\mathcal{H}(U) = \{holomorphism\ functions\ U \to \mathbb{C}\}$, $\mathcal{G}(U) = \{holomorphic\ function : U \to \mathbb{C}^*\}$, $exp : \mathcal{H}(U) \to \mathcal{G}(U)$, $exp : \mathcal{H} \to \mathcal{G}$.
$Im(exp)$ is not a sheaf, $g_i \in \mathcal{H}(U_i), f_i = exp(g_i)$, $\mathcal{U} = \cup_i \mathcal{U}_i$, $f_i \in \mathcal{G}(U_i)$, $f_{i_{U_i \cap U_j}} = f_j|_{U_i \cap U_j}$ does not implies $\exists g \in \mathcal{H}(U)$ such that $exp(g)|_{U_i} = f_i$.

2.5 Inductive limit

Definition 2.7. *We are given a set I of indexes. $\forall i \in I$, a group E_i such that*
— *I is partially ordered and $\forall i, j \in I$, $\exists k \in I$, $i \leqslant k$, $j \leqslant k$;*
— *$\forall i, j \in I$, $i \geqslant j$, $\phi_{i,j} : E_i \to E_j$ group homomorphism. If $i \leqslant j \leqslant k$;*

$$\begin{tikzcd} E_i \arrow[r, "\phi_{ij}"] \arrow[d, "\phi_{ik}"'] & \phi_{jk} E_j \arrow[dl] \\ E_k & \end{tikzcd}$$

— *$\phi_{ii} = Id_{ii}$.*
$\varinjlim E_i$ inductive limit is a group. The homomorphism $E_k \to \varinjlim_i E_i$ $\forall k$ such that if $k \leqslant j$,

$$\begin{tikzcd} E_k \arrow[r, "\phi"] \arrow[d] & E_j \arrow[dl, "\phi_j"] \\ \varinjlim_i E_i & \end{tikzcd}$$

(Universal property)
If G is a group and $\forall i$, $\psi_i : E_i \to G$ group homomorphism such that $\forall i \leqslant j$,

$$\begin{tikzcd} E_i \arrow[r, "\phi_{ij}"] \arrow[d, "\phi_i"'] & E_j \arrow[dl, "\psi_j"] \\ G & \end{tikzcd}$$

then there exists a unique factorization $\psi : \varinjlim_i E_i \to G$, $\forall i$,

$$\begin{tikzcd} E_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & G \arrow[dl, "\psi"] \\ \varinjlim_i E_i & \end{tikzcd}$$

Proposition 2.8. *$\varinjlim_i E_i$ exists and is unique up to unique isomorphism.*

Proof. — Uniqueness: standard argument (up)
existence: Set $\coprod_{i\in I} E_i$, $\sim$ *$x, y \in \coprod_i E_i \Rightarrow x \in E_{i_0}$ and $u \in E_{j_0}$.*
$x \sim y$ if $\exists k_0 \geqslant i_0$ and j_0 such that $\phi_{i_0k_0}(x) = \phi_{j_0k_0}(g)$. Take $k_2 \geqslant k_0, k_1$,

$$\begin{tikzcd} E_{i_0} \arrow[dr] & & E_{j_0} \arrow[dl] \arrow[dr] & & E_{k_0} \arrow[dl] \\ & E_{k_0} \arrow[dr] & & E_{k_1} \arrow[dl] & \\ & & E_{k_2} & & \end{tikzcd}$$

$\coprod_i E_i/\sim$, one can check $\phi_k : E_k \to \coprod_i E_i \to \coprod E_i/\sim$, $\{\phi_k : E_k \to \coprod_i E_i/\sim\}_k$ is an inductive limit. □

Example 2.9. *X topological space, $\mathcal{F}$ pre sheaf on X, $x_0 \in X$, $I = \{$open subsets of X containing $x_0\}$, $U, V \in I$, $U \leqslant V$ if $V \subseteq U$, $E_U = \mathcal{F}(U)$, $U \leqslant V \Rightarrow V \subseteq U$,*

$$\begin{tikzcd} \phi_U|_V : E_U \arrow[r] \arrow[d, equal] & E_V \arrow[d, equal] \\ \mathcal{F}(U) \arrow[r, "\rho_{UV}"] & \mathcal{F}(V) \end{tikzcd}$$

$U, V \in I \Rightarrow U \leqslant U \cap U$ and $V \leqslant U \cap V$.

Definition 2.10. *$\mathcal{F}_{x_0} := \varinjlim x_0 \in U\mathcal{F}(\mathcal{U})$: the stalk of $\mathcal{F}$ at x_0.*

Example 2.11. *$\mathcal{F} = C^0_X(\mathbb{R})$, $x_0 \in X$, $\mathcal{F}_{x_0} = \{[f] | f \in C^0(U, \mathbb{R}), x_0 \in U\}$*
— $[f] = [g]$ if $g = f$ on some open neighborhood of x_0;
— $s \in \mathcal{F}(U)$, $x_0 \in U$, denote by s_{x_0} the image of s in $\mathcal{F}_{x_0}$.

Lemma 2.12. *Let $\mathcal{F}$ be a sheaf on X then $\forall U \subseteq X$, the map*

$$\begin{array}{ccc} \mathcal{F}(U) & \to & \coprod_{x\in U}\mathcal{F}_x \\ s & \mapsto & (s_x)_{x\in U} \end{array}$$

is injective

Proof. Let $s, t \in \mathcal{F}(U)$ such that $s_x = t_x$, $\forall x \in U$. For any $x \in U$, $s_x = t_x$ mean, $\exists x \in V_x$ such that $s|_{V_x} = t|_{V_x}$. $V = \cup_{x\in U} V_x \Rightarrow s = t$. □

Proposition 2.13. *Let $\mathcal{G} : \mathcal{F} \to \mathcal{G}$ morphism of sheaves. For any $x \in X$, suppose that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism then ϕ is an isomorphism.*

Proof. Definition of ϕ_x
$\phi : \mathcal{F} \to \mathcal{G}$ morphism of presheaves, $x_0 \in X$, $\phi_{x_0} : \mathcal{F}_{x_0} \to \mathcal{G}_{x_0}$, $\forall x_0 \in U$, $x_0 \in V \subseteq U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[d, "can"] \arrow[dr, "\rho_{UV}"] & \mathcal{G}(U) \arrow[l, "\phi(U)"'] \arrow[r, "can"] \arrow[dr, "can"] & \mathcal{G}_{x_0} \\ \mathcal{F}_{x_0} & \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \arrow[u] \end{tikzcd}$$

$s \in \mathcal{F}_{x_0} \to \mathcal{G}_{x_0} \Rightarrow \exists x_0 \in U,\ t \in \mathcal{F}(U)$, such that $s = t_{x_0}$.

$$\varinjlim lim E_i = \coprod E_i/sim$$

$t \in \mathcal{F}(U)$, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ $(\phi(U)(t))_{x_0}$ such that $s = t_{x_0}$.
$U \subseteq X$, $\phi(U) : \mathcal{F}(U) \simeq \mathcal{G}(U)$

1. $\phi(U)$ injective

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \prod_{x\in C}\mathcal{F}_x \arrow[r] & \prod_{x\in C}\mathcal{G}_x \end{tikzcd}$$

   $t \in \mathcal{G}(U)$, $\forall x \in U$, $t_x = \phi_x(s(x))$, $s(x) \in \mathcal{F}_x$, $\exists s_x \in \mathcal{F}(U_x)$ depending on x such that $sx = s(x)$.
   $t_x = \phi_x(s)$, $t_x = \phi(V_x)(s)_x = \phi_x(s_x)$ in $\mathcal{G}(V_x)$. t and $\phi(V_x)(s)$ some stalk at x then $\exists x \in W_x \subseteq V_x$ on which $t|_{W_x} = s|_{W_x}$. $U = \cup_{x\in U} W_x$ Be careful $s = s^x$ depend on x, $(s^x|W_x)_x$, $s^x|_{W_x\cap W_y} s^x|_{W_x\cap W_y}$ glueing condition $\Rightarrow \exists s \in \mathcal{F}(U)$, $s|W_x = s^x$, $\forall x$, $\phi(U)(s)_x = t_x$, $\forall x \in C \Rightarrow \phi(U)(s) = t \Rightarrow \phi(U)$ surjective.

□

2.6 Sheaf associated to a presheaf

Let $\mathcal{F}$ be a presheaf on X of opposite sets.

Proposition 2.14. *There is a unique pair $\mathcal{F}^+$ sheaf on X and $\theta : \mathcal{F} \to \mathcal{F}^+$ a morphism of presheave such that $\forall \mathcal{F} \to \mathcal{G}$ from $\mathcal{F}$ to a sheaf $\mathcal{G}$ has an unique factorization*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, "\psi^+"'] & \end{tikzcd}$$

$$\forall x \in X,\ \mathcal{F}_x \simeq \mathcal{F}_x^+$$

Definition 2.15.

1. *If $f : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves then $Im(\phi)$ is the sheaf associated to the pre sheaf $U \mapsto Im(\phi(U))$.*
2. *If $\mathcal{F} \subset \mathcal{G}$ sub sheaf then $\mathcal{G}/\mathcal{F}$ = sheaf associated to the pre sheaf $U \to \mathcal{G}(U)/\mathcal{F}(U)$;*
3. *We say that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is surjective if $Im(\phi) = \mathcal{G}$ ($\Leftrightarrow \phi_x : \mathcal{F}_x \twoheadrightarrow \mathcal{G}_x$ $\forall x \in X$) in $\underline{sh}_X$, ϕ is surjective if and only if ϕ is an epimorphism.*
4. *(Exact sequence) $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ mean $ker(\psi) = Im(\phi)$*

$$\Leftrightarrow \forall x \in X,\ \mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \ \text{exact;}$$

5. *$f : X \to Y$ continuou, $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on $Y \Rightarrow f_*\mathcal{F}$ direct image of $\mathcal{F}$ by f.*
6. *$f_*\mathcal{F} : U \subseteq Y \to \mathcal{F}(\phi^{-1}(V))$, $U \subseteq X$ sheaf associated $U \to \varinjlim_{f(U) \subset W, W \text{ open in } Y} \mathcal{G}(W)$. Presheaf but in general not a sheave*

$$\forall x \in X,\ (f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$$